A quantum exactly solvable non-linear oscillator related with the isotonic oscillator

J.F. Cariñena† [a)], A.M. Perelomov‡ [b)], M.F. Rañada† [c)],
and M. Santander§ [d)]

† *Departamento de Física Teórica and IUMA, Facultad de Ciencias*
Universidad de Zaragoza, 50009 Zaragoza, Spain

‡ *Departamento de Física Teórica, Facultad de Ciencias*
Universidad de Zaragoza, 50009 Zaragoza, Spain

§ *Departamento de Física Teórica, Facultad de Ciencias*
Universidad de Valladolid, 47011 Valladolid, Spain

Abstract

A nonpolynomial one-dimensional quantum potential representing an oscillator, that can be considered as placed in the middle between the harmonic oscillator and the isotonic oscillator (harmonic oscillator with a centripetal barrier), is studied. First the general case, that depends of a parameter a, is considered and then a particular case is studied with great detail. It is proven that it is Schrödinger solvable and then the wave functions Ψ_n and the energies E_n of the bound states are explicitly obtained. Finally it is proven that the solutions determine a family of orthogonal polynomials $\mathcal{P}_n(x)$ related with the Hermite polynomials and such that: (i) Every $\mathcal{P}_n$ is a linear combination of three Hermite polynomials, and (ii) They are orthogonal with respect to a new measure obtained by modifying the classic Hermite measure.

Keywords: Quantum integrable systems. Isotonic oscillator. Nonpolynomial potentials.

Running title: A quantum exactly solvable non-linear oscillator.

PACS numbers: 03.65.-w, 03.65.Ge, 02.30.Gp

AMS classification: 81Q05, 81U15, 33C45 Dirac, Klein-Gordon

[b)] On leave of absence from Institute for Theoretical and Experimental Physics, 117259 Moscow, Russia

E-mail : [a)] `jfc@unizar.es` ; [b)] `perelomo@dftuz.unizar.es` ; [c)] `mfran@unizar.es` ; [d)] `msn@fta.uva.es`.

1 Introduction

It is well known that, in quantum mechanics, the family of Schrödinger solvable potentials is very restricted and also that exact solvability is a very delicate property (see Ref. [1] for a review and Refs. [2]–[11] for some recent papers on this matter). In fact, in most of cases, the addition of a small perturbation to a quantum solvable system breaks this property and leads to potentials that must be analyzed by the use of perturbative methods, variational formalisms or numerical techniques. The most interesting and best known system inside this small family is the harmonic oscillator whose energy spectrum consists of an infinite set of equidistant energy levels. Many other oscillators, as for example harmonic oscillators perturbed by a term containing a fourth or a sixth power in the coordinate, have been extensively studied but making use of the above mentioned techniques. Nevertheless, it is known the existence of another oscillator, the so-called isotonic oscillator, that is exactly solvable and is endowed with many interesting properties.

The main aim of this paper is to present an study of a new nonpolynomial one-dimensional quantum potential representing an oscillator, that can be considered as placed in the middle between the harmonic oscillator and the isotonic oscillator. We will prove that is exactly solvable and that the energy spectrum and the wave functions have properties closely related with those characterising the harmonic oscillator.

In more detail, the plan of the article is as follows: In Sec. II we recall the main characteristics of the isotonic oscillator In Sec. III we present a new potential depending of a parameter a; first we study the general case and then we solve the particular case $a^2 = 1/2$ obtaining the wave functions and energy spectrum. Sec. IV is devoted to analyze a family of orthogonal polynomials and to study the relation with the Hermite polynomials. Finally, in Sec. V we make some final comments.

2 The Isotonic oscillator

The following potential

$$U_{\rm Isot}(x) = U_0(x) + U_g(x) = (\frac{1}{2})\,\omega^2\,x^2 + (\frac{1}{2})\,\frac{g}{x^2}\,, \quad g > 0\,,$$

representing an harmonic oscillator with a centripetal barrier, is known as the isotonic oscillator [12, 13]. It is important because is endowed with properties closely related with those of the harmonic oscillator. At the classical level it leads to periodic solutions with the same period (isochronous potential [14]) and at the quantum level it is Schrödinger solvable, the Hamiltonian is factorizable [15] and the energy spectrum is equidistant. Moreover, it is related with supersymmetric quantum mechanics [16]. The two-dimensional version is superintegrable and corresponds to the so-called Smorodinsky-Winternitz system and the centripetal term relates it with the Calogero-Moser system. In addition to all these theoretical properties, this particular nonlinear oscillator is important in quantum optics and in the theory of coherent states [17, 18].

The classical Lagrange equation, that is given by

$$\ddot{x} + \omega^2 x - \frac{g}{x^3} = 0 \,,$$

is a particular case of the Pinney-Ermakov equation [19]. It can be exactly solved and the solution is given by

$$x = \frac{1}{\omega A} \sqrt{(\omega^2 A^4 - g)\,\sin^2(\omega t + \phi) + g} \,.$$

showing explicitly the above mentioned periodicity. At the quantum level it is convenient to write g in the way $g = m(m+1)$ (the constant g should be greater than $-1/4$ and m can be any real number but we will always take it as non-negative) so that the Schrödinger equation takes the form

$$\frac{d^2}{dx^2}\,\Psi - \Big[\omega^2 x^2 + \frac{m(m+1)}{x^2}\Big]\Psi + 2E\Psi = 0 \,.$$

where we assume $\hbar = 1$ for easy of notation. The simplest solution, representing the ground state, is

$$\Psi_0 = N_0\, x^{1+m}\, \exp(-\frac{1}{2}\omega\, x^2) \,, \quad E_0 = (\frac{3}{2} + m)\,\omega \,.$$

and the all the other wave functions and eigenenergies are given by

$$\Psi_{2n} = N_{2n}\, x^{1+m}\, P_{2n}(x)\, \exp(-\frac{1}{2}\omega\, x^2) \,, \quad E_{2n} = E_0 + 2\,n\,\omega \,,$$

where $P_{2n}(x)$ is a polynomial of order $2n$ with only even powers of x and N_{2n} denotes the normalization constant. The energy spectrum is equidistant since

$$E_{2n+2} = E_{2n} + 2\,\omega \,, \quad n = 0, 1, 2, \ldots$$

Nevertheless, the height ΔE of the energy steps is twice that of the simple harmonic oscillator U_0. In fact, it seems as if half of the levels (those with an odd number of nodes) have disappeared.

3 A new solvable potential

We now turn our attention to the study of the one-dimensional quantum system described by the following potential

$$U_{0a}(x) = U_0(x) + U_a(x) = (\frac{1}{2})\,\Big[\omega^2\, x^2 + 2g_a\,\frac{x^2 - a^2}{(x^2 + a^2)^2}\Big], \quad g_a > 0 \,,$$

where a is a positive real parameter. The reason for this particular algebraic expression is that the new additional term can be written as the sum of two centripetal barriers in the complex plane

$$2\,\frac{x^2 - a^2}{(x^2 + a^2)^2} = \frac{1}{(x + \mathrm{i}\, a)^2} + \frac{1}{(x - \mathrm{i}\, a)^2} \,,$$

so it is a rational potential with two imaginary poles symmetric with respect the origin. Actually, if g_a remains constant, then when a goes to zero and to ∞ the potential $U_{0a}(x)$ converges to the isotonic and the harmonic oscillator, respectively.

The first important property is that if the coefficient g_a is not arbitrary but given by $g_a = 2\,\omega\,a^2(1+2\,\omega\,a^2)$ then the Schrödinger equation, that takes the form

$$\frac{d^2}{dx^2}\,\Psi - \Big[\omega^2\,x^2 + 4\,\omega\,a^2(1+2\,\omega\,a^2)\,\frac{x^2-a^2}{(x^2+a^2)^2}\,\Big]\Psi + 2E\Psi = 0\,, \tag{1}$$

admits the following solution

$$\Psi_0 = \frac{N_0}{(a^2+x^2)^{2\,\omega\,a^2}}\;\exp(-\frac{1}{2}\omega\,x^2)\,,\quad E_0 = \frac{1}{2}\,\omega - (2\,\omega\,a)^2\,. \tag{2}$$

It represents the ground level (Ψ_0 has no nodes) and it clearly shows that when $a \to 0$ then the corresponding wave function and ground level energy of the linear oscillator is obtained. We also note that the above expression $g_a = 2\,\omega\,a^2(1+2\,\omega\,a^2)$ can be alternatively written as $g_a = m_a(m_a+1)$ with $m_a = 2\,\omega\,a^2$.

In the following we will focus our study on the particular case $a^2 = 1/2$, $m_a = \omega$ (see Figure 1) represented by the equation

$$\frac{d^2}{dx^2}\,\Psi - \left[x^2 + 8\,\frac{2x^2-1}{(2x^2+1)^2}\right]\Psi + 2E\Psi = 0 \tag{3}$$

so that Ψ_0 and E_0 become

$$\Psi_0(x) = \frac{N_0}{1+2x^2}\;\exp(-\frac{1}{2}x^2)\,,\quad E_0 = -\,\frac{3}{2}\,, \tag{4}$$

where we have assumed $\omega = 1$ for easy of notation.

As a method for obtaining all the other eigenstates we assume that the functions $\Psi(x)$ can be factorized in the form

$$\Psi(x) = F(x)\,\Psi_0(x)$$

and, as the lowest energy is $E_0 = -3/2$, it seems appropriate to write the general energy E as follows

$$E = -\,\frac{3}{2} + e\,.$$

Then the Schrödinger equation (3) leads to the following equation for the function $F(x)$

$$(1+2x^2)\,F'' - 2x\,(5+2x^2)\,F' + 2\,e\,(1+2x^2)\,F = 0\,. \tag{5}$$

Since the origin $x = 0$ is an ordinary point we expect an analytic solution F with a power series expansion convergent in the interval $(-R, R)$ with the radius of convergence R given by $R = 1/\sqrt{2}$

$$F(x) = \sum_{n=0}^{\infty} p_n x^n = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \ldots$$

that when replaced in (5) leads to

$$2(p_2+ep_0)+[6p_3+2(e-5)p_1]\,x+\sum_{m=0}^{\infty}\Bigg[(m+4)(m+3)\,p_{m+4}$$
$$+2[\,(m+2)(m-4)+e\,]\,p_{m+2}+4(e-m\,)\,p_m\Bigg]\,x^{m+2}=0\,. \tag{6}$$

Therefore, we first obtain the following two relations for the first coefficients p_0, p_2 and p_1, p_3

$$p_2+e\,p_0=0\,,\qquad 3p_3+(e-5)\,p_1=0\,,$$

and then the general recursion relation

$$(m+4)(m+3)\,p_{m+4}+2[\,(m+2)(m-4)+e\,]\,p_{m+2}+\,4(e-m)\,p_m=0\,,\quad m=0,1,2,\dots\,.$$

Hence the recurrence relation involve three different terms (p_{m+4} depends of both p_{m+2} and p_m) and is constrained by the two first relations that are not included in the general rule. In any case even coefficients are related among themselves and the same is true for odd coefficients. The general solution will be obtained by fixing the values of p_0 and p_1, i.e. the value $F(0)$ and $F'(0)$. The solution determined by $F(0)=1$, $F'(0)=0$ is even (only contains even powers) while the one determined by $F(0)=0$, $F'(0)=1$ only contains odd powers of x. Actually the expressions of the first coefficients, in terms of p_0 and p_1, are given by

$$\begin{array}{ll}
p_2=-\,e\,p_0 & p_8=\frac{2^4}{8\,!}(e-50)(e-6)(e-4)e\,p_0\\
p_4=\frac{2^2}{4\,!}(e-10)e\,p_0 & p_{10}=-\frac{2^5}{10\,!}(e-82)(e-8)(e-6)(e-4)e\,p_0\\
p_6=-\frac{2^3}{6\,!}(e-26)(e-4)e\,p_0 & \dots\dots\dots\dots\dots\dots
\end{array}$$

and

$$\begin{array}{ll}
p_3=-\frac{2}{3\,!}(e-5)\,p_1 & p_9=\frac{2^4}{9\,!}(e-65)(e-7)(e-5)(e-3)\,p_1\\
p_5=\frac{2^2}{5\,!}(e-17)(e-3)\,p_1 & p_{11}=-\frac{2^5}{11\,!}(e-101)(e-9)(e-7)(e-5)(e-3)\,p_1\\
p_7=-\frac{2^3}{7\,!}(e-37)(e-5)(e-3)\,p_1 & \dots\dots\dots\dots\dots\dots
\end{array}$$

In the particular case in which e is an even integer number $e=2k$, with $k=3,4,5,\dots$, all the coefficients $p_{2(k+r)}$ vanish and the series reduces to a polynomial of degree k in powers of x^2, $P_{2k}(x)$. Similarly, when e is an odd integer number $e=2k+1$, with $k=2,3,4,\dots$, all the coefficients $p_{2(k+r)+1}$ with $r>1$ vanish, and the series reduces to a polynomial P_{2k+1} with only odd powers of x. The first polynomial solutions are:

$$\begin{array}{llll}
P_0 & =1 & & \\
P_4 & =1-4x^2-4x^4 & P_3 & =x+\frac{2}{3}x^3\\
P_6 & =1-6x^2-4x^4+\frac{8}{3}x^6 & P_5 & =x-\frac{4}{5}x^5\\
P_8 & =1-8x^2-\frac{8}{3}x^4+\frac{32}{5}x^6-\frac{16}{15}x^8 & P_7 & =x-\frac{2}{3}x^3-\frac{4}{3}x^5+\frac{8}{21}x^7\\
P_{10} & =1-10x^2+\frac{32}{3}x^6-\frac{80}{21}x^8+\frac{32}{105}x^{10} & P_9 & =x-\frac{4}{3}x^3-\frac{8}{5}x^5+\frac{16}{15}x^7-\frac{16}{135}x^9
\end{array}$$

Two important properties are: First, there are no polynomial solutions of degree $k=1$ and $k=2$ and, second, all of them have two complex conjugate roots, so that P_3 has an unique real

zero, P_4 has two real zeros and, in the general case, P_{2k} and P_{2k+1} with $k > 1$ have $2k-2$ and $2k-1$ real zeros respectively.

Another remarkable property is that when the polynomials P_n are expressed as linear combination of Hermite polynomials then only a finite number of terms are different from zero. We have obtained the following relations for the first cases

$$\begin{array}{llll}
P_4 & = -4H_0 - 4H_2 - \frac{1}{4}H_4 & P_3 & = H_1 + \frac{1}{12}H_3 \\
P_6 & = 3H_2 + H_4 + \frac{1}{24}H_6 & P_5 & = -H_1 - \frac{1}{2}H_3 - \frac{1}{40}H_5 \\
P_8 & = -\frac{2}{3}H_4 - \frac{2}{15}H_6 - \frac{1}{240}H_8 & P_7 & = \frac{1}{3}H_3 + \frac{1}{12}H_5 + \frac{1}{336}H_7 \\
P_{10} & = \frac{1}{12}H_6 + \frac{1}{84}H_8 + \frac{1}{3360}H_{10} & P_9 & = -\frac{1}{20}H_5 - \frac{1}{120}H_7 - \frac{1}{4320}H_9
\end{array}$$

These particular relations clearly suggest that each polynomial P_{2k} can be written as a linear combination of H_{2k} and the two previous even Hermite polynomials, H_{2k-2} and H_{2k-4}; similarly the odd polynomial P_{2k+1} appears as a linear combination of only H_{2k+1}, H_{2k-1} and H_{2k-3}.

4 Wave functions and orthogonality relations

The differential equation (5) is not in selfadjoint form because, if we denote by a_0, a_1 and a_2 the three coefficients of the equation, we have

$$a_0 = 1 + 2x^2\,, \quad a_1 = -2x\,(5 + 2x^2)\,, \quad \frac{da_0}{dx} \neq a_1\,.$$

However, use can be made of an integrating factor $\mu(x)$ such that

$$\frac{d}{dx}[\mu(x)a_0(x)] = \mu(x)a_1(x)\,.$$

so that $\mu(x)$ is given by

$$\mu(x) = (\frac{1}{a_0})\, e^{\int (a_1/a_0)\, dx} = \frac{e^{-x^2}}{(1+2\,x^2)^3}\,,$$

in such a way that (5) becomes

$$\frac{d}{dx}\Big[\, p(x)\,\frac{dF}{dx}\,\Big] + 2e\, r(x)\, F = 0\,,$$

where the two functions $p = p(x)$ and $r = r(x)$ are given by

$$p(x) = \frac{e^{-x^2}}{(1+2\,x^2)^2}\,, \qquad r(x) = \frac{e^{-x^2}}{(1+2\,x^2)^2}\,.$$

This equation, with the appropriate conditions for the behaviour of the solutions at the end points, constitutes a Sturm–Liouville (S-L) problem, defined in the real line $\mathbb{R}$. According to this, the eigenfunctions of the S-L problem are orthogonal with respect to the weight function

$r = e^{-x^2}/(1+2\,x^2)^2$, and, in particular, the polynomial solutions P_m, $m = 0, 3, 4, \ldots$, of the differential equation (5), satisfy the orthogonality conditions

$$\int_{-\infty}^{\infty} P_m(x)\, P_n(x)\, \frac{e^{-x^2}}{(1+2\,x^2)^2}\; dx = 0\,, \quad m \neq n\,.$$

Let us now return to the Eq. (5) and suppose for F the following factorization

$$F' = (1+2x^2)\, G\,.$$

Then we arrive after some calculus (we omit the details) to the following equation for the function G

$$G'' - 2x\, G' + 2(e-3)\, G = 0\,,$$

that means that the derivative $P_n'(x)$ of the polynomial $P_n(x)$ must satisfy

$$P_n' = (1+2x^2)\, H_{n-3}\,, \quad e = n\,, \quad n = 3, 4, 5, \ldots$$

(up to a multiplicative constant). At this point we recall the following two properties of the Hermite polynomials

$$\begin{array}{ll} (i) & 2xH_m = H_{m+1} + 2mH_{m-1} \\ (ii) & H_m' = 2mH_{m-1} \end{array}$$

Then making use of (i) we arrive to

$$P_n' = \frac{1}{2}\left[H_{n-1} + 4(n-2)\, H_{n-3} + 4(n-3)(n-4)\, H_{n-5}\right],$$

and making use of (ii) and then integrating we obtain

$$P_n = \frac{1}{4n}\left[H_n + (4n)\, H_{n-2} + (4n)(n-3)\, H_{n-4}\right].$$

It seems convenient to multiply P_n by $4n$ and introduce the new family of polynomials $\mathcal{P}_n$ defined in the form

$$\mathcal{P}_n = H_n + 4\,n\, H_{n-2} + 4\,n\,(n-3)\, H_{n-4}\,, \quad n = 3, 4, 5, \ldots \tag{7}$$

so that the coefficient of H_n (the dominant term in the expression of $\mathcal{P}_n$) reduces to unity.

Proposition 1 *The following equality holds*

$$\frac{2n\, P_n\, e^{-x^2}}{(1+2x^2)^2} = -\frac{d}{dx}\left[\frac{H_{n-3}}{1+2x^2}\, e^{-x^2}\right], \quad n = 3, 4, 5, \ldots$$

Proof: This statement is proven just by making the calculus.

Now we can write

$$\int_{-\infty}^{\infty} \frac{\mathcal{P}_m(x)\,\mathcal{P}_n(x)}{(1+2\,x^2)^2}\; e^{-x^2}\; dx = -\,(16nm)\int_{-\infty}^{\infty} \frac{1}{2m}\frac{d}{dx}\Big[\frac{H_{m-3}}{1+2x^2}\,e^{-x^2}\Big]\,P_n(x)\; dx$$

and integrating by parts we arrive to

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\mathcal{P}_m(x)\,\mathcal{P}_n(x)}{(1+2\,x^2)^2}\; e^{-x^2}\; dx &= (8n)\int_{-\infty}^{\infty}\Big[\frac{H_{m-3}}{1+2x^2}\,e^{-x^2}\Big]\,P_n'(x)\; dx \\ &= (8n)\int_{-\infty}^{\infty} H_{m-3}\,H_{n-3}\,e^{-x^2}\; dx \\ &= \delta_{mn}\,(8n)\,[\,2^{n-3}\,(n-3)\,!\,\sqrt{\pi}\,] \qquad (8)\end{aligned}$$

So, we relate the problem of normalization with the standard problem of Hermite polynomials. Hence the orthogonality conditions for the family $\mathcal{P}_n(x)$ read:

$$\int_{-\infty}^{\infty} \mathcal{P}_m(x)\,\mathcal{P}_n(x)\,r(x)\,dx = \delta_{mn}\int_{-\infty}^{\infty} \frac{[\,\mathcal{P}_n(x)\,]^2}{(1+2\,x^2)^2}\; e^{-x^2}\; dx = k_n\,(\,2^n\,n\,!\,\sqrt{\pi}\,)\,,$$

where the proportionality constant k_n is given by

$$k_n = \frac{1}{(n-1)(n-2)}\,.$$

See Figures 2a and 2b for the plot of some of these polynomials.

If we define the *P-Hermite functions* Ψ_m by

$$\Psi_m(x) = \frac{\mathcal{P}_m(x)}{(1+2\,x^2)}\; e^{-(1/2)\,x^2}\,, \quad m = 0,3,4,\ldots$$

then the above property admits the following alternative form:

$$\int_{-\infty}^{\infty} \Psi_m(x)\,\Psi_n(x)\; dx = 0\,, \quad m \neq n\,.$$

In summary, the eigenfunctions corresponding to the lowest energy levels are:

$$\begin{aligned}\Psi_0(x) &= N_0\,\frac{\mathcal{P}_0(x)}{(1+2\,x^2)}\; e^{-(1/2)\,x^2}\,, \quad E_0 = -\,3/2 \\ \Psi_3(x) &= N_3\,\frac{\mathcal{P}_3(x)}{(1+2\,x^2)}\; e^{-(1/2)\,x^2}\,, \quad E_3 = 3/2 = -\,3/2+3 \\ \Psi_4(x) &= N_4\,\frac{\mathcal{P}_4(x)}{(1+2\,x^2)}\; e^{-(1/2)\,x^2}\,, \quad E_4 = 5/2 = -\,3/2+4 \\ \Psi_5(x) &= N_5\,\frac{\mathcal{P}_5(x)}{(1+2\,x^2)}\; e^{-(1/2)\,x^2}\,, \quad E_5 = 7/2 = -\,3/2+5\end{aligned}$$

where the normalization constant is

$$N_k = \left[\frac{(k-1)(k-2)}{2^k \, k! \, \sqrt{\pi}} \right]^{1/2} .$$

The first three wave functions, $\Psi_0(x)$, $\Psi_3(x)$, $\Psi_4(x)$, together with the corresponding wave functions of the harmonic oscillator $\Phi_0(x)$, $\Phi_1(x)$, $\Phi_2(x)$, are plotted in Figures 3, 4 and 5.

The energy E_0 of the ground state $\Psi_0(x)$ has been singled out of all the other values and moved into the smaller value $E_0 = -3/2$. The rest of the energy spectrum consists, as in the pure harmonic case, of an infinite set of equidistant energy levels

$$E_{n+1} = E_n + 1 \,, \quad n = 3, 4, 5, \ldots$$

Let us close this section with two comments on the new family of polynomials we have obtained. First the definition (7) of $\mathcal{P}_n$ as a linear combination of *only three* Hermite polynomials can be considered as a particular case of a situation known as a *special linear combinations of orthogonal polynomials* (see [20, 21] and references therein). Finally, let us mention that taking into account the "Rodrigues formula" for the Hermite polynomials

$$H_n(x) = (-1)^n \, e^{x^2} \, \frac{d}{dx^n} \, e^{-x^2}$$

we obtain

$$\mathcal{P}_n(x) = (-1)^n \, e^{x^2} \left[\frac{d^n}{dx^n} + 4n \, \frac{d^{n-2}}{dx^{n-2}} + 4 \, n \, (n-3) \, \frac{d^{n-4}}{dx^{n-4}} \right] e^{-x^2} \,,$$

that must be considered as the "Rodrigues formula" for this new family of orthogonal polynomials.

5 Final comments and outlook

We have proved that the potential $U_{0a}(x)$ can be exactly solved in the particular case $a^2 = 1/2$ and also that it possesses two very remarkable properties. First, the fundamental level Ψ_0 has an energy E_0 that is lower than in the pure harmonic case and, in a sense, is isolated of all the other values. Second, the rest of the energy spectrum is endowed with the equidistance property. Concerning the general case, with an arbitrary value for the parameter a, we have only obtained the expression for the fundamental level $(\Psi_0(a), E_0(a))$. The resolution of the general case remains as an open question that deserves be studied. Finally let us also mention that the analysis of this potential using the supersymmetric quantum mechanics as an approach also seems an interesting matter to be studied.

Acknowledgments

The paper was finished when one of authors (A.P.) was the guest of Max-Planck-Institut für Mathematik (Bonn); he would like to thank the staff of MPI for hospitality. We also thank

M. Alfaro for discussions on the properties of orthogonal polynomials. Support of projects MTM-2006-10531, FPA-2003-02948, E23/1 (DGA), MTM-2005-09183, and VA-013C05 is acknowledged.

References

[1] M.A. Olshanetsky and A.M. Perelomov, "Quantum integrable systems related to Lie algebras", Phys. Rep. **94**, 313-404 (1983).

[2] R. Dutt, A. Gangopadhyaya, C. Rasinariu and U. Sukhatme, "New solvable singular potentials", J. Phys. A **34**, 4129–4142 (2001).

[3] R. Koc, M. Koca and E. Körcük, "A new class of quasi-exactly solvable potentials with a position-dependent mass", J. Phys. A **35**, L527–L530 (2002).

[4] S.A.S. Ahmed and B.C. Borah, "Construction of solvable quantum mechanical potentials", J. Phys. A **36**, 10071–10082 (2003).

[5] F. Calogero, "On the quantization of two other nonlinear harmonic oscillators", Phys. Lett. A **319**, 240–245 (2003).

[6] K. Andriopoulos and P.G.L. Leach, "Some group theoretical aspects of nonlinear quantal oscillators", J. Nonlinear Math. Phys. **12**, suppl. 1, 32–42 (2005).

[7] C.M. Bender and M. Monou, "New quasi-exactly solvable sextic polynomial potentials", J. Phys. A **38**, 2179–2187 (2005).

[8] B. Bagchi, A. Banerjee, C. Quesne and V.M. Tkachuk, "Deformed shape invariance and exactly solvable Hamiltonians with position-dependent effective mass", J. Phys. A **38**, 2929-2945 (2005).

[9] M. Znojil, "Perturbation method for non-square Hamiltonians and its application to polynomial oscillators", Phys. Lett. A **341**, 67–80 (2005)

[10] J.F. Cariñena, M.F. Rañada and M. Santander, "A quantum exactly solvable nonlinear oscillator with quasi-harmonic behaviour", Ann. Phys. **322**, 434–459 (2007).

[11] S. Cruz y Cruz, J. Negro and L.M. Nieto, "Classical and quantum position-dependent mass harmonic oscillators", Phys. Lett. A **369**, 400-406 (2007).

[12] Y. Weissman and J. Jortner, "The isotonic oscillator", Phys. Lett. A **70**, 177–179 (1979).

[13] D. Zhu, "A new potential with the spectrum of an isotonic oscillator", J. Phys. A **20**, 4331–4336 (1987).

[14] O.A. Chalykh and A.P. Veselov, "A remark on rational isochronous potentials", J. Nonlinear Math. Phys. **12**, suppl. 1, 179–183 (2005).

[15] L. Infeld and T.E. Hull, "The factorization method", Rev. Mod. Phys. **23**, 21–68 (1951).

[16] J. Casahorran, "On a novel supersymmetric connections between harmonic and isotonic oscillators", Physica A **217**, 429-439 (1995).

[17] J.S. Wang, T.K. Liu and M.S. Zhan, "Nonclassical properties of even and odd generalized coherent states for an isotonic oscillator", J. Optics B (Quantum Semiclass. Opt.) **2**, 758–763 (2000).

[18] K. Thirulogasanthar and N. Saad, "Coherent states associated with the wavefunctions and the spectrum of the isotonic oscillator", J. Phys. A **37**, 4567–4577 (2004).

[19] E. Pinney, "The nonlinear differential equation $y'' + p(x)y' + cy^{-3} = 0$", Proc. Amer. Math. Soc. **1**, 681 (1950).

[20] G. Szegö, *Orthogonal polynomials*, 4th ed., Amer. Math. Soc. Colloquium Publications, Vol. 23 (Amer. Math. Soc., Providence, R.I., 1975).

[21] Z. Grinshpun, "Special linear combinations of orthogonal polynomials", J. Math. Anal. Appl. **299**, 1-18 (2004).

Figure Captions

- FIGURE 1. Plot of the potential $U_{0a}(x)$ for $\omega = 1$ as a function of x for $a^2 = 1/2$ (continuous line) together with the plot of the harmonic oscillator (dash line). The main difference lies in the form of the minimum that is much deeper in the $U_{0a}(x)$ case than in the linear case. Nevertheless, for great values of $|x|$ the two functions have rather the same form.

- FIGURE 2. (2a) Polynomials $\mathcal{P}_3$ (dash line) and $\mathcal{P}_4$ (continuous line). P_3 has a unique real zero at the origin and P_4 has two real zeros (symmetric with respect the origin). (2b) Polynomials $\mathcal{P}_5$ (dash line) and $\mathcal{P}_6$ (continuous line). P_5 has three real zeros (the origin and two other placed symmetric) and P_6 has four zeros (two positive and two negative)..

- FIGURE 3. Wave function Ψ_0 (continuous line) and wave function Φ_0 of the harmonic oscillator (dash line).

- FIGURE 4. Wave function Ψ_3 (continuous line) and wave function Φ_1 of the harmonic oscillator (dash line).

- FIGURE 5. Wave function Ψ_4 (continuous line) and wave function Φ_2 of the harmonic oscillator (dash line).

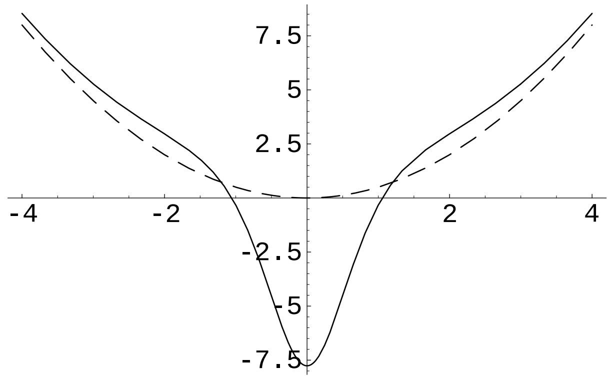

Figure 1: Plot of the potential $U_{0a}(x)$ for $\omega = 1$ as a function of x for $a^2 = 1/2$ (continuous line) together with the plot of the harmonic oscillator (dash line). The main difference lies in the form of the minimum that is much deeper in the $U_{0a}(x)$ case than in the linear case. Nevertheless, for great values of $|x|$ the two functions have rather the same form.

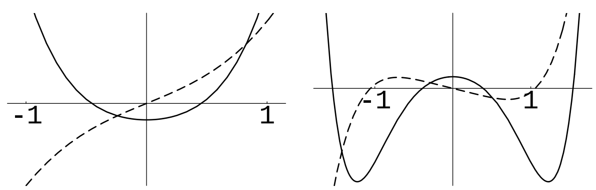

Figure 2: (2a) Polynomials $\mathcal{P}_3$ (dash line) and $\mathcal{P}_4$ (continuous line). P_3 has a unique real zero at the origin and P_4 has two real zeros (symmetric with respect the origin). (2b) Polynomials $\mathcal{P}_5$ (dash line) and $\mathcal{P}_6$ (continuous line). P_5 has three real zeros (the origin and two other placed symmetric) and P_6 has four zeros (two positive and two negative).

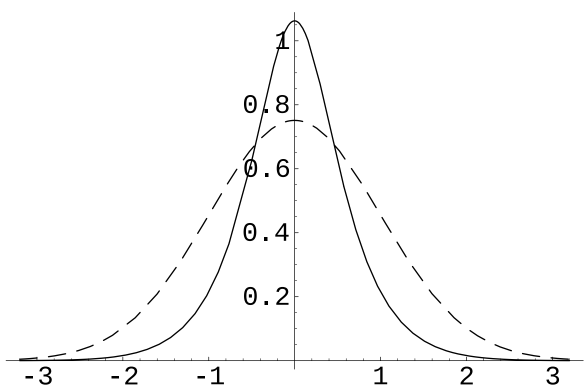

Figure 3: Wave function Ψ_0 (continuous line) and wave function Φ_0 of the harmonic oscillator (dash line).

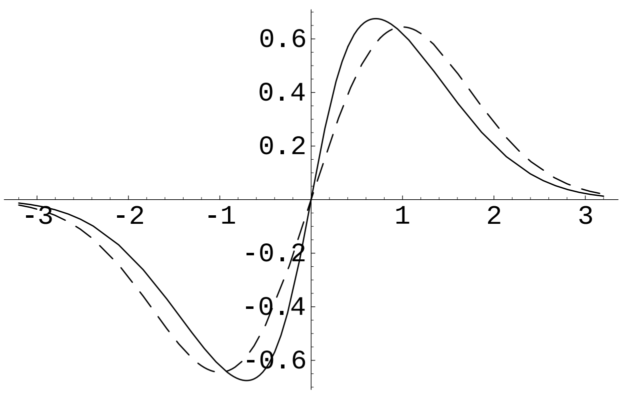

Figure 4: Wave function Ψ_3 (continuous line) and wave function Φ_1 of the harmonic oscillator (dash line).

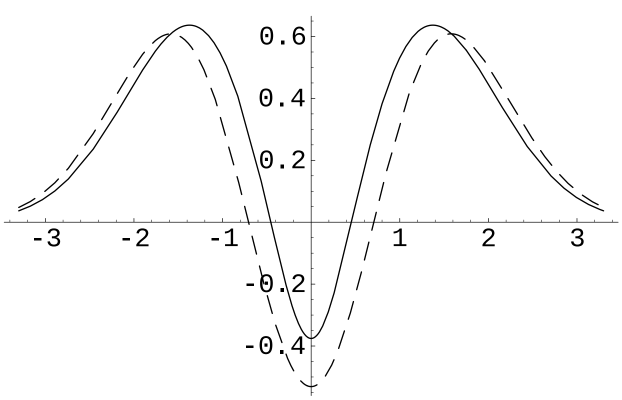

Figure 5: Wave function Ψ_4 (continuous line) and wave function Φ_2 of the harmonic oscillator (dash line).